E.O. PYASKOVSKY
Ivanovo State University,
School of Mathematics and IT

ARITHMETIC OF QUATERNIONS

Introduction

- The rapid and fruitful development of complex analysis in the 19th century stimulated the interest of mathematicians to solve the next task: to find a new kind of numbers similar in their properties to the complex ones, but containing not one, but two imaginary units. However, this work was unsuccessful.
- A new kind of numbers was discovered by the Irish mathematician William Hamilton in 1843, and it contained not two, as it had been expected, but three imaginary units. Hamilton called these numbers “quaternions”.
- Despite the unusual properties of the new numbers (their non-commutativity with respect to multiplication), this model quickly brought practical benefits.

History of the Problem

Let us recall, that the quaternion algebra $\boldsymbol{H}$ can be considered as a vector space over the field $\boldsymbol{R}$ of real numbers with the basis $\{1,\ i,\ j,\ k\}$ in which the multiplication is given by the formulas:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Therefore, every element (quaternion) x of algebra $\boldsymbol{H}$ can be written in the form

$$x = \alpha + \beta i + \gamma j + \delta k, \text{где } \alpha, \beta, \gamma, \delta \in \boldsymbol{R}.$$

It is known, that $\boldsymbol{H}$ is a body (or division algebra).

The quaternion $x = \alpha + \beta i + \gamma j + \delta k$ is called integer if either all its components α, β, γ, δ are integer, or they are all half odd integer numbers.

It is easy to show that the set of whole quaternions is a non-commutative ring.

Proof

Using the arithmetic properties of the ring of integer quaternions, one can obtain a well-known number-theoretic result – the Lagrange theorem that says that any natural number can be represented as the sum of four squares of integers.

As a corollary of this theorem, one more interesting result of number can be given, which dates back to Diophantus, formulated by Fermat and proved by Euler: every prime number of the form $4n + 1$ can be represented as the sum of two squares of integers.

Conclusion

Taking into consideration the properties of integer quaternions it is possible to get a stronger result about some natural numbers in the form of the sum of three squares of integers.